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On the Optimal System and Series Solutions of Fifth-Order Fujimoto-Watanabe Equations

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Abstract

This paper investigates the two fifth-order Fujimoto-Watanabe equations from the perspective of the group theoretic approach. We identify the reduced equations that lead to the solutions of these high order equations. Furthermore, the corresponding solutions are found by power series due to their nonlinear characteristics. As a result, the findings of the study demonstrate the convergence of solutions for such models and identifies the travelling wave solutions.

Keywords: lie symmetries; fifth-order equations; convergence; power series.

1 Introduction

In a remarkable paper, Fujimoto-Watanabe [5], explored two fifth-order equations

$$A: u_t = u^5 u_{xxxxx} + 5u^4 \big(u_x u_{xxxx} + 2u_{xx} u_{xxx}\big), \tag{1}$$

$$B: u_t = u^5 u_{xxxxx} + 5u^4 \left(u_x u_{xxxx} + \frac{1}{2} u_{xx} u_{xxx}\right) + \frac{15}{4} u^3 u_x^2 u_{xxx}. \tag{2}$$

The full classification of fifth-order evolution equations [5] is too long and tedious, that it was not found, unlike the case of third-order evolution equations. As a special consideration, (1) and (2) may be mapped onto the Sawada-Kotera [26] and Kaup [14] equations via a Schwarzian derivative [24], and via the inverse scattering transform technique at zero spectral parameter [25]. Due to these properties, the above equations have applications in fluid dynamics, ion-acoustic waves in plasmas and nonlinear shallow water wave phenomena, to name a few. These equations are also important to study from other perspectives, as interesting notions may apply, such as the inverse scattering technique. The third-order class has been investigated [6].

As an instrument in the study of all possible types of differential equations, transformations are the heart and soul of analytical solution techniques. Furthermore, transformative methods may sometimes be used in conjunction with numerical approaches. Explicitly, transformations simplify the equation, either by creating an easier form of the equation, or through combining variables, and thereby decreasing the number of variables. The only challenge, of course, is the calculation of such transformations. However, these days, symmetry methods are the most reliable method to generate transformations of *most* differential equations.

Under the umbrella of solutions to differential equations, we have the concept that a symmetry of a differential equation is a transformation (or mapping) of its solution manifold into itself [2]. In other words, these transformations map any solution of the given equation into another solution of the same equation [3], where invariant or similarity solutions are those that map into themselves [13]. Of special interest, are point symmetries with regards to double reductions [9], invariance analysis [10], optimal systems [11], difference equations [8], delay equations [16], and quintic nonlinear equations [27].

In terms of partial differential equations, the invariance under a one-parameter Lie group of transformations enacts a reduction of the number of independent variables, while with ordinary differential equations, the order may be reduced. The infinitesimal generator of the transformation is applied so that solutions arise constructively. We can easily list more applications of symmetries; namely: the determination of conservation laws [17] and its admitted solutions (periodic solutions [7], Noetherian related ones [4] or a standard group analysis solution [20]), linearization techniques [18], finding integrating factors [19] and canonical coordinates.

In this paper, we explore Lie symmetry reductions of this family of equations, which are naturally of high order, and therefore must be solved using special techniques. Such a technique that works well, is the power series approach.

The organisation of the study is provided next. In the next section, we display the symmetry invariant properties admitted by the first equation, with its commutator, adjoint and group specifications. Section 3 has the associated symmetry details for the second equation. In Section 4, we have a demonstration of the convergence of solutions for our fifth-order reduced equations. The conclusion is given in Section 5.

2 Case A

Let us invoke a Lie group of (one-parameter) infinitesimal ($\varepsilon \ll 1$) mappings, for x, t, u to $x + \varepsilon\,\hat{\xi}$, $t + \varepsilon\,\hat{\tau}$ and $u \to u + \varepsilon\,\hat{\phi}$, respectively and with $\hat{\phi}$, $\hat{\xi}$ and $\hat{\tau}$ as functions of (x, t, u).

Now, a symmetry vector field is $X = \hat{\phi}\,\partial_u + \hat{\xi}\,\partial_x + \hat{\tau}\,\partial_t$, where the unknown coefficient functions in (x, t, u) are given by the standard Lie symmetry determining condition. Omitting the detailed calculations (see [12] for a simple discussion or for further examples: a classical scenario [22], a fractional context [15] and a travelling wave case [28]).

The Lie point symmetries of (1) are four, viz,

$$X_1 = \partial_x, \qquad X_2 = \partial_t, \qquad X_3 = u\partial_u + x\partial_x, \qquad X_4 = 5t\partial_t + x\partial_x, \qquad X_5 = 2ux\partial_u + x^2\partial_x. \tag{3}$$

The Lie brackets of (3) are given in Table 1 where $[X_j, X_k]$ is the commutator given by

$$[X_j, X_k] = X_j X_k - X_k X_j.$$

Table 1: Lie brackets.

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_1	X_1	$2X_3$
X_2	0	0	0	$5X_2$	0
X_3	$-X_1$	0	0	0	X_5
X_4	$-X_1$	$-5X_2$	0	0	X_5
X_5	$-2X_3$	0	$-X_5$	$-X_5$	0

The one-parameter groups M_j $(j = 1, \ldots, 4)$ of the symmetries X_j, are

$$M_1 : (x, t, u) \mapsto (x + \varepsilon, t, u), \tag{4}$$
$$M_2 : (x, t, u) \mapsto (x, t + \varepsilon, u), \tag{5}$$
$$M_3 : (x, t, u) \mapsto (e^{\varepsilon}x, t, e^{\varepsilon}u), \tag{6}$$
$$M_4 : (x, t, u) \mapsto (e^{\varepsilon}x, e^{5\varepsilon}t, u), \tag{7}$$
$$M_5 : (x, t, u) \mapsto \left(\frac{x}{-1 + x\varepsilon}, t, \frac{u}{(-1 + x\varepsilon)^2}\right). \tag{8}$$

The adjoints are given in Table 2 and we define the optimal system of one-dimensional subalgebras for Cases A. The approach for obtaining the one-dimensional optimal system is taken from [21] and involves the formula

$$\mathrm{Ad}\Big(\exp(\epsilon X_j)\Big)X_k = X_k - \epsilon[X_j, X_k] + \frac{1}{2}\,\epsilon^2\Big[X_j, [X_j, X_k]\Big] - \ldots.$$

Table 2: Adjoint's.

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$X_3 - \varepsilon X_1$	$X_4 - \varepsilon X_1$	$\varepsilon^2 X_1 - 2\varepsilon X_3 + X_5$
X_2	X_1	X_2	X_3	$X_4 - 5\varepsilon X_2$	X_5
X_3	$e^{\varepsilon} X_1$	X_2	X_3	X_4	$e^{-\varepsilon} X_5$
X_4	$e^{\varepsilon} X_1$	$e^{5\varepsilon} X_2$	X_3	X_4	$e^{-\varepsilon} X_5$
X_5	$\varepsilon^2 X_5 + 2\varepsilon X_3 + X_1$	X_2	$\varepsilon X_5 + X_3$	$\varepsilon X_5 + X_4$	X_5

Here, we consider,

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5, \tag{9}$$

where the a_is are arbitrary constants, and simplify this by the applications of adjoint maps.

Suppose firstly that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\mathrm{Ad}\Big(\exp\Big(\frac{1}{2a_5}\Big)X_1\Big)$, we get

$$X^{'} = a_1^1X_1 + a_2X_2 + a_4X_4 + a_5X_5,$$

$a_5 \neq 0$, and the scalar a_1^1 depends on a_1, a_4, a_5. Now acting on $X^{'}$ by $\mathrm{Ad}\Big(\exp\Big(\frac{a_2}{5a_4}\Big)X_2\Big)$ results in $X^{''} = a_1^1X_1 + a_4X_4 + a_5X_5$, $a_4 \neq 0$. Next, $\mathrm{Ad}\Big(\exp\big(c_3X_3\big)\Big)$ on $X^{''}$ yields,

$$X_4, \quad aX_4 \pm X_1, \quad aX_4 \pm X_5, \quad aX_4 \pm X_5 \pm X_1. \tag{10}$$

If $a_5 = 0$, then $X = a_1X_1 + a_2X_2 + X_3 + a_4X_4$. We act on X by $\mathrm{Ad}\Big(\exp\Big(\frac{a_2}{5a_4}\Big)X_2\Big)$ and obtain

$$X^{'} = a_1X_1 + X_3 + a_4X_4,$$

$a_4 \neq 0$. Further acting on $X^{'}$ by $\mathrm{Ad}\Big(\exp\big(c_2X_3\big)\Big)$ gives

$$X_3 + aX_4, \quad X_3 + aX_4 \pm X_1. \tag{11}$$

If $a_4 = 0$, then we act on $X = a_1X_1 + a_2X_2 + X_3$ by $\mathrm{Ad}\Big(\exp\big(a_1X_1\big)\Big)$ and obtain $X^{'} = X_3 + a_2X_2$, and further acting on $X^{'}$ by $\mathrm{Ad}\Big(\exp\big(c_2X_4\big)\Big)$ gives rise to

$$X_3, \quad X_3 \pm X_2. \tag{12}$$

If $a_5 \neq 0$, $a_4 = 0$, then $X^{'} = a_1^1X_1 + a_2X_2 + a_5X_5$. Acting on $X^{'}$ by $\mathrm{Ad}\Big(\exp\big(c_2X_3\big)\Big)$ one obtains

$$bX_2 \pm X_5, \quad bX_2 \pm X_5 \pm X_1. \tag{13}$$

If $a_3 = 0$, $a_4 = 1$, then we act on $X = a_1X_1 + a_2X_2 + X_4 + a_5X_5$ by $\mathrm{Ad}\Big(\exp\Big(\frac{a_2}{5}\Big)X_2\Big)$ and get $X^{'} = a_1X_1 + X_4 + a_5X_5$. Further acting on $X^{'}$ by $\mathrm{Ad}\Big(\exp\big(c_3X_3\big)\Big)$ gives

$$X_4, \quad X_4 \pm X_1, \quad X_4 \pm X_5, \quad X_4 \pm X_5 \pm X_1. \tag{14}$$

Assume now $a_4 = a_3 = 0$, $a_5 = 1$, then we act on $X = a_1X_1 + a_2X_2 + X_5$ by $\text{Ad}\Big(\exp\big(c_1X_3\big)\Big)$ which yields X_5, $X_5 \pm X_1$, $X_5 \pm X_2$, $X_5 \pm X_1 \pm X_2$.

Now assume that $a_5 = a_4 = a_3 = 0$, $a_1 = 1$, we act on $X = X_1 + a_2X_2$ by $\text{Ad}\Big(\exp\big(c_1X_3\big)\Big)$ which gives rise to X_1, $X_1 \pm X_2$. Next, suppose that $a_1 = a_5 = a_4 = a_3 = 0$, $a_2 = 1$ that yields X_2. Now, the above discussion gives rise to the optimal system of one-dimensional subalgebras for this case,

$$aX_4 + \delta X_5 + \epsilon X_1, \quad X_2, \quad aX_4 + X_3 + \epsilon X_1, \quad bX_2 + \delta X_5 + \epsilon X_1,$$
$$X_3 + \epsilon X_2, \quad X_4 + \epsilon X_1, \quad X_1 + \epsilon X_2,$$

where $a(\neq 0)$, b are constants, $\delta = \pm 1$ and $\epsilon = 0, \pm 1$.

2.1 Symmetry invariant reductions

In this subsection, we apply a Lie reduction technique, whereby zero-order invariants transform the order of the given partial differential equation. This procedure results in reduced equations that are difficult to solve by most analytical techniques. We instead apply series to determine explicit solutions for (1).

A reduction of the equation by the symmetry vectors $X_1 + X_2$ $(\epsilon = 1)$ produces the ordinary differential equation

$$y'(z)\Big(1 + 5y(z)^4 y''''(z)\Big) + y(z)^4\Big(5y''(z)y'''(z) + y(z)y^{(5)}(z)\Big) = 0, \tag{15}$$

where $z = -x + t$ and $u(x,t) = y(z)$. Next, the power series

$$y(z) = \sum_{r_1=0}^{\infty} a_{r_1} z^{r_1}, \tag{16}$$

is entered into (15). We see that a_0, a_1, a_2, a_3, a_4 are the arbitrary constants and that a solution reads as

$$\begin{aligned} y(z) = {} & a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \frac{\big(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\big)z^5}{120a_0^5} \\ & + \frac{z^6}{360a_0^5}\Bigg(-90a_3^2a_0^4 - 240a_2a_4a_0^4 - 120a_1a_2a_3a_0^3 - 240a_1^2a_4a_0^3 - a_2 \\ & - \frac{5a_1\big(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\big)}{a_0}\Bigg) + \ldots. \end{aligned} \tag{17}$$

Hence a solution for (1) may be obtained by reversing the transformation, namely,

$$\begin{aligned} u(x,t) = {} & a_0 + a_1\left(-x+t\right) + a_2\left(-x+t\right)^2 + a_3\left(-x+t\right)^3 \\ & + a_4\left(-x+t\right)^4 + \frac{\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)\left(-x+t\right)^5}{120a_0^5} \\ & + \frac{\left(-x+t\right)^6}{360a_0^5}\Bigg(-90a_3^2a_0^4 - 240a_2a_4a_0^4 - 120a_1a_2a_3a_0^3 - 240a_1^2a_4a_0^3 - a_2 \\ & - \frac{5a_1\left(-60a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right)}{a_0}\Bigg) + \ldots. \end{aligned} \tag{18}$$

The question arises of whether such a series solution converges or not. We answer this question in Section 4.

Suppose we consider the next reduction from the list of optimal one-dimensional subalgebras, that is, $aX_4 + \delta X_5 + \epsilon X_1$. This symmetry produces the invariant,

$$z = t\exp\left(-\frac{10a\tan^{-1}\left(\frac{a}{\sqrt{4\delta\epsilon-a^2}}+\frac{2\delta x}{\sqrt{4\delta\epsilon-a^2}}\right)}{\sqrt{4\delta\epsilon-a^2}}\right),$$

where

$$u(x,t) = \left(ax+\delta x^2+\epsilon\right)\exp\left(-\frac{2a\tan^{-1}\left(\frac{a+2\delta x}{\sqrt{4\delta\epsilon-a^2}}\right)}{\sqrt{4\delta\epsilon-a^2}}\right)y\left(z\right). \tag{19}$$

In this instance, (1) reduces to the high-order nonlinear ordinary differential equation

$$\begin{aligned}
&16a\delta\epsilon\big(3a^2+4\delta\epsilon\big)y(z)^6+y'(z)\\
&\quad+125a^3\delta^2y(z)^4\Bigg(84\big(10a^2+\delta\epsilon\big)y'(z)^2+50a^2\delta^2y''(z)\big(18y''(z)+5zy'''(z)\big)\\
&\qquad\qquad+5y'(z)\Big(12\big(48a^2+8\epsilon\big)y''(z)+5a^2\delta+5a^2z\big(48y'''(z)+5zy'''(z)\big)\Big)\Bigg)\\
&\quad+5azy(z)^5\Bigg(4\big(756a^4+327a^2\delta\epsilon+16\delta^2\epsilon^2\big)y'(z)+25a^2\delta\Big(84\big(10a^2+8\epsilon\big)y''(z)\\
&\qquad\qquad+5z\Big(4\big(48a^2+8\epsilon\big)y'''(z)\,5a^2z\big(12y''''(z)+zy^{(z)}(z)\big)\Big)\Big)\Bigg)=0.
\end{aligned} \tag{20}$$

By the same procedure, that is the power series (16), we find the series solution $(\epsilon = 1)$,

$$\begin{aligned}
y(z) &= a_0-16a\left(3a^2+4\right)a_0^6z-2\left(3a^2+4\right)\Big(-60480a^6a_0^{11}-27312a^4a_0^{11}-2816a^2a_0^{11}\Big)z^2\\
&\quad-\frac{8}{3}\Bigg(5779200a^7\left(3a^2+4\right)^2a_0^{16}+1405440a^5\times\left(3a^2+4\right)^2a_0^{16}+81920a^3\left(3a^2+4\right)^2a_0^{16}\\
&\quad-60060a^5\left(3a^2+4\right)\times\Big(-60480a^6a_0^{11}-27312a^4a_0^{11}-2816a^2a_0^{11}\Big)a_0^5\\
&\quad-8592a^3\times\left(3a^2+4\right)\left(-60480a^6a_0^{11}-27312a^4a_0^{11}-2816a^2a_0^{11}\right)a_0^5\\
&\quad-256a\left(3a^2+4\right)\left(-60480a^6a_0^{11}-27312a^4a_0^{11}-2816a^2a_0^{11}\right)\times a_0^5\Bigg)z^3+\ldots,
\end{aligned} \tag{21}$$

which may be transformed into a solution in $u(x,t)$, using (19), for equation (1).

Similarly, a reduction of equation (1) by the symmetry vector X_2 gives

$$y(z)y^{(5)}(z)+5y^{(4)}(z)y'(z)+5y^{(3)}(z)y''(z)=0,$$

where $u = y(z)$, $z = x$, and the series solution is

$$y(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \frac{(-a_2 a_3 - 2a_1 a_4)\, z^5}{2a_0} + \frac{\left(-3a_3^2 - 8a_2 a_4 - \frac{6a_1(-a_2a_3-2a_1a_4)}{a_0}\right) z^6}{12a_0} + \frac{\left(-\frac{7\left(-3a_3^2-8a_2a_4-\frac{6a_1(-a_2a_3-2a_1a_4)}{a_0}\right)a_1}{4a_0} - 12a_3a_4 - \frac{8a_2(-a_2a_3-2a_1a_4)}{a_0}\right) z^7}{21a_0} + \ldots . \tag{22}$$

Repeating this idea, with the symmetry $X_4 + \epsilon X_1$, we have the transformation $z = \dfrac{t}{(x+\epsilon)^5}$ where $u = y(z)$. Hence, the reduced equation is $(\epsilon = 1)$

$$\begin{aligned} &y'(z)\left(625z^3y(z)^4\left(576y''(z) + 5z\left(48y^{(3)}(z) + 5zy^{(4)}(z)\right)\right) + 15120zy(z)^5 + 1\right) \\ &+ 105000z^2y(z)^4y'(z)^2 + 625z^2y(z)^4\left(10z^2y''(z)\left(18y''(z) + 5zy^{(3)}(z)\right)\right. \\ &\left. + y(z)\left(5z^3y^{(5)}(z) + 60z^2y^{(4)}(z) + 168y''(z) + 192zy^{(3)}(z)\right)\right) = 0, \end{aligned} \tag{23}$$

with series solution,

$$\begin{aligned} y(z) &= a_0 + a_1 z - 7560a_1a_0^5z^2 - 280\left(215a_0^4a_1^2 - 2162160a_0^{10}a_1\right)z^3 \\ &- 120\left(-38064600a_1^2a_0^9 - 813960\left(215a_0^4a_1^2 - 2162160a_0^{10}a_1\right)a_0^5 + 1190a_1^3a_0^3\right)z^4 + \ldots . \end{aligned} \tag{24}$$

Next, the symmetry combination $X_3 + \epsilon X_2$ results in the invariant $z = t - \epsilon\log(x)$, $u(x,t) = xy(z)$, so that the reduced equation is

$$\begin{aligned} &y'(z)\left(1 + 4\epsilon y(z)^5 + 5y(z)^4\left(-3\epsilon y''(z) + \epsilon^5 y''''(z)\right)\right) \\ &+ \epsilon^3 y(z)^4\left(10\epsilon^2 y''(z)y'''(z) + y(z)\left(-5y'''(z) + \epsilon^2 y^{(5)}(z)\right)\right) = 0. \end{aligned} \tag{25}$$

Here, we find the solution $(\epsilon = 1)$,

$$\begin{aligned} y(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \\ &+ \frac{\left(-4a_1a_0^5 + 30a_3a_0^5 + 30a_1a_2a_0^4 - 120a_2a_3a_0^4 - 120a_1a_4a_0^4 - a_1\right) z^5}{120a_0^5} + \ldots . \end{aligned} \tag{26}$$

In the penultimate reduction, we use the vector $bX_2 + \delta X_5 + \epsilon X_1$, so that

$$z = \frac{\sqrt{\delta}t\sqrt{\epsilon} - b\tan^{-1}\left(\frac{\sqrt{\delta}x}{\sqrt{\epsilon}}\right)}{\sqrt{\delta}\sqrt{\epsilon}},$$

and $u(x,t) = \left(\delta x^2 + \epsilon\right) y(z)$. Eq. (1) becomes

$$\begin{aligned} &y'(z)\left(1 + 64b\delta^2\epsilon^2 y(z)^5 + 5y(z)^4\left(12b^3\delta\epsilon y''(z) + b^5 y'''(z)\right)\right) \\ &+ b^3 y(z)^4\left(10b^2 y''(z)y'''(z) + y(z)\left(20\delta\epsilon y'''(z) + b^2 y^{(5)}(z)\right)\right) = 0, \end{aligned} \tag{27}$$

with solution ($\epsilon = 1$),

$$y(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \frac{z^5\Big(- 120a_0^4 a_1 a_2 b^3\delta - 120a_0^5 a_3 b^3\delta - 120a_0^4 a_2 a_3 b^5 - 120a_0^4 a_1 a_4 b^5 - 64a_0^5 a_1 b\delta^2 - a_1\Big)}{120a_0^5 b^5} + \dots . \tag{28}$$

Finally, the symmetry $aX_4+X_3+\epsilon X_1$, with $z = t(ax+x+\epsilon)^{-\frac{5a}{a+1}}$, and $u(x,t) = (ax+x+\epsilon)^{\frac{1}{a+1}}y(z)$, produces the reduction

$$\begin{aligned}
&4a\big(1+4a+a^2-6a^3\big)y(z)^6 + y'(z) \\
&+ 125a^3z^2y(z)^4\Bigg(21\big(-1-21+40a^2\big)y'(z)^2 + 50a^2z^2y''(z)\big(18y''(z)+5zy'''(z)\big) \\
&\qquad + 5zy'(z)\Big(\big(-3-6a+576a^2\big)y''(z) + 5a^2z\big(48y'''(z)+5zy''''(z)\big)\Big)\Bigg) \\
&+ 5azy(z)^5\Bigg(\big(4+16a-311a^2-654a^3+3024a^4\big)y'(z) + 25a^2z\Bigg(21\big(-1-2a+40a^2\big)y'''(z) \\
&\qquad + 5z\Big(\big(-1-2a+192a^2\big)y'''(z) + 5a^2z\big(12y''''(z)+zy^{(5)}\big)\Big)\Bigg)\Bigg) = 0,
\end{aligned} \tag{29}$$

with solution

$$\begin{aligned}
y(z) = {}& a_0 + a_1 z + \frac{1}{2}\Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)z^2 \\
& - \frac{4}{3}\Bigg(45150a_0^4a_1^2a^5 - 5490a_0^4a_1^2a^4 - 2585a_0^4a_1^2a^3 + 160a_0^4a_1^2a^2 + 40a_0^4a_1^2a \\
&\quad + 30030a_0^5\Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)a^5 \\
&\quad - 2148a_0^5 \times \Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)a^4 \\
&\quad - 1042a_0^5\Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)a^3 \\
&\quad + 32a_0^5\Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)a^2 \\
&\quad + 8a_0^5\Big(- 15120a^5a_1a_0^5 + 3414a^4a_1a_0^5 + 1531a^3a_1a_0^5 - 176a^2a_1a_0^5 - 44aa_1a_0^5\Big)a\Bigg)z^3 \\
& + \dots .
\end{aligned} \tag{30}$$

3 Case B

We now turn to Case B with a similar analysis. The Lie point symmetries of Eq. (2) are

$$X_1 = \partial_x, \qquad X_2 = \partial_t, \qquad X_3 = 5t\partial_t - u\partial_u, \tag{31}$$

and the Lie commutator brackets of (31) are displayed in Table 3.

Table 3: Lie brackets.

$[,]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$5X_2$
X_3	0	$-5X_2$	0

Thus, we have the relevant M_j of this case are

$$M_1 : (x, t, u) \mapsto (x + \varepsilon, t, u), \tag{32}$$
$$M_2 : (x, t, u) \mapsto (x, t + \varepsilon, u), \tag{33}$$
$$M_3 : (x, t, u) \mapsto (x, e^{5\varepsilon} t, e^{\varepsilon} u). \tag{34}$$

The adjoints are given in Table 4.

Table 4: Adjoint's.

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - 5\varepsilon X_2$
X_3	X_1	$e^{5\varepsilon} X_2$	X_3

We consider

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3, \tag{35}$$

where the a_is are arbitrary constants, and as before simplify it by adjoint maps.

Suppose first that $a_3 \neq 0$, then we can assume that $a_3 = 1$. If we act on X by $\text{Ad}\bigg(\exp\left(\frac{a_2}{5}\right) X_2\bigg)$ we get

$$X_3 + a X_1. \tag{36}$$

If $a_3 = 0$, $a_2 = 1$, then we act on $X = a_1 X_1 + X_2$ by $\text{Ad}\Big(\exp\left(c_1 X_3\right)\Big)$ results in X_2, $X_2 \pm X_1$. Next suppose that $a_3 = a_2 = 0$, $a_1 = 1$ gives X_1. Now, the above procedure gives for this case, the one dimensional subalgebras

$$X_1, \qquad X_3 + a X_1, \qquad X_2 + \epsilon X_1,$$

where a is a constant and $\epsilon = 0, \pm 1$.

3.1 Symmetry invariant reductions

A reduction by X_1 gives

$$y'(t) = 0, \tag{37}$$

with $u(x,t)=y(t)$. If one solves this, we get

$$y(t)=c_1, \tag{38}$$

which is a trivial constant solution.

Reducing the equation (2) by $X_2+\epsilon X_1$ gives

$$y'(z)\Big(2\epsilon^5-15\epsilon y(z)^3y'''(z)+10y(z)^4y''''(z)\Big)+y(z)^4\Big(5y''(z)y'''(z)+2y(z)y'''''(z)\Big)=0, \tag{39}$$

where $z=\dfrac{-x+t\epsilon}{\epsilon}$ and $u(x,t)=y(z)$. The power series solution we find is $(\epsilon=1)$

$$y(z)=a_0+a_1z+a_2z^2+a_3z^3+\frac{\left(-30a_2a_3a_0^4+45a_1a_3a_0^3-a_1\right)z^4}{120a_0^4a_1}+\dots. \tag{40}$$

Reducing the equation by X_3+aX_1 gives the ordinary differential equation

$$\begin{aligned}
&17y(z)^6+2a^5y'(z)-375z^2y(z)^3y'(z)\Big(43y'(z)+5z\big(18y''(z)+5zy'''(z)\big)\Big)\\
&\quad+25zy(z)^4\Big(4095zy'(z)^2+5z\big(-3ae^{\frac{x}{a}}+25z^2y''(z)\big)\big(18y''(z)+5zy'''(z)\big)\\
&\qquad\qquad+y'(z)\big(-132a+16275z^2y''(z)+9375z^3y'''(z)+1250z^4y'''(z)\big)\Big)\\
&\quad+5y(z)^5\Big(-3a+5960zy'(z)+41625z^2y''(z)+47875z^3y'''(z)\\
&\qquad\qquad+15000z^4y''''(z)+1250z^5y'''''(z)\Big)=0,
\end{aligned} \tag{41}$$

where $z=e^{\frac{5x}{a}}t$ and $u=e^{\frac{x}{a}}y(z)$. Hence a solution is

$$\begin{aligned}
y(z)&=a_0+\frac{1}{2}\big(15a_0^5-17a_0^6\big)z+\frac{1}{4}\Bigg(\frac{3375}{2}a_0^4\big(15a_0^5-17a_0^6\big)-14951a_0^5\big(15a_0^5-17a_0^6\big)\Bigg)z^2\\
&+\frac{1}{6}\Bigg(-118988\Big(\frac{3375}{2}a_0^4\big(15a_0^5-17a_0^6\big)-14951a_0^5\big(15a_0^5-17a_0^6\big)\Big)a_0^5\\
&\qquad+\frac{20175}{4}\Big(\frac{3375}{2}a_0^4\big(15a_0^5-17a_0^6\big)-14951a_0^5\big(15a_0^5-17a_0^6\big)\Big)a_0^4\\
&\qquad-\frac{125815}{2}\big(15a_0^5-17a_0^6\big)^2a_0^4+\frac{29475}{4}\big(15a_0^5-17a_0^6\big)^2a_0^3\Bigg)z^3+\dots.
\end{aligned} \tag{42}$$

4 Convergence of Series

In this section, we demonstrate the convergence properties [1] associated with the above solution types. Suppose we observe Eq. (15) with (16) substituted into it, viz.

$$\begin{aligned}\sum_{r_1=0}^{\infty}\Bigg(&(r_1+1)a_{r_1+1}+\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Bigg(\sum_{r_6=0}^{r_5}a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_1-r_2+5}(r_1-r_2+4)\\&\times(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)\Bigg)\\&+\Big(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2}(r_1-2r_2+3)(r_1-2r_2+2)\\&\qquad\times(r_1-2r_2+1)a_{r_1-2r_2+3}\Big)\\&+\Big(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4}(r_1-2r_2+4)(r_1-2r_2+3)\\&\qquad\times(r_1-2r_2+2)(r_1-2r_2+1)\Big)\Bigg)z^{r_1}=0.\end{aligned} \tag{43}$$

Therefore, by extensive manipulation, the coefficients are generated by

$$\begin{aligned}a_{r_1+5}=&-\frac{(r_1+1)a_{r_1+1}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\\&-\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Bigg(5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}(r_2+2)(r_2+1)a_{r_2+2}\\&\qquad\times\frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\\&\qquad+5a_{r_5}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}a_{r_2+1}a_{r_1-2r_2+4}\\&\qquad\times\frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)}\Bigg)\\&-\sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}a_{r_6}a_{r_5-r_6}a_{r_4-r_5}a_{r_3-r_4}a_{r_2-r_3}\\&\qquad\times\frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5(r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)},\end{aligned} \tag{44}$$

for $r_1 \geq 0$, such that (17) reads as

$$y(z)=a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+\sum_{r_1=0}^{\infty}a_{r_1+5}z^{r_1+5}. \tag{45}$$

Next, from Eq. (44) we obtain

$$
\begin{aligned}
|a_{r_1+5}| \leq M\Bigg(|a_{r_1+1}| &+ \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Big(|a_{r_5}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_2+2}||a_{r_1-2r_2+3}| \\
&+ |a_{r_5}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_2+1}||a_{r_1-2r_2+4}|\Big) \\
&+ \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}|a_{r_6}||a_{r_5-r_6}||a_{r_4-r_5}||a_{r_3-r_4}||a_{r_2-r_3}||a_{r_1-r_2+5}|\Bigg),
\end{aligned}
\tag{46}
$$

where $r_1 = 0, 1, 2, \ldots$, and $M = \dfrac{1}{a_0^5}$.

Consider the series $\hat{\mu} = R(z) = \sum_{r_1=0}^{\infty} p_{r_1} z^{r_1}$ where

$$
p_k = |a_k|, \qquad k = 0, \ldots, 4, \tag{47}
$$

and

$$
\begin{aligned}
p_{r_1+5} = M\Bigg(p_{r_1+1} &+ \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\Big(p_{r_5}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_2+2}p_{r_1-2r_2+3} \\
&+ p_{r_5}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_2+1}p_{r_1-2r_2+4}\Big) \\
&+ \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5}p_{r_6}p_{r_5-r_6}p_{r_4-r_5}p_{r_3-r_4}p_{r_2-r_3}p_{r_1-r_2+5}\Bigg).
\end{aligned}
\tag{48}
$$

Hence,

$$
|a_{r_1}| \leq p_{r_1}, \qquad r_1 = 0, 1, 2, \ldots. \tag{49}
$$

We are now required to show that $\hat{\mu}$ converges. The series $\hat{\mu}$ is a majorant series of equation (45) so that:

$$
\begin{aligned}
R(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \sum_{r_1=0}^{\infty} p_{r_1+5} z^{r_1+5} \\
&= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + M\Big[R^5 + R\gamma(z) + \nu(z)\Big],
\end{aligned}
\tag{50}
$$

where $\nu(z) = \theta(z) - p_0^5$, with $\theta(z)$ and $\gamma(z)$ are polynomial expressions such that every term contains a power of at least one in z. Then, let

$$
H(z, \hat{\mu}) = \hat{\mu} - p_0 - p_1 z - p_2 z^2 - M\Big[R^5 + R\gamma(z) + \nu(z)\Big], \tag{51}
$$

be an implicit function such that $H(0, p_0) = 0$ and $H_{\hat{\mu}}(0, p_0) = -5Mp_0^4 + 1 \neq 0$. We now invoke the implicit function theorem [23], seeing that $\hat{\mu}$ is convergent and analytic in the neighbourhood of $(0, p_0)$ in the plane, and with a positive radius. We conclude that (45) is convergent in the same neighbourhood.

Hence, solution (45) is expressed as,

$$\begin{aligned} y(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \\ &+ \sum_{r_1=0}^{\infty} \Bigg(- \frac{(r_1+1)a_{r_1+1}}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\ &- \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4} \Big(5a_{r_5} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} (r_2+2)(r_2+1) a_{r_2+2} \\ &\times \frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\ &+ 5a_{r_5} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} a_{r_2+1} a_{r_1-2r_2+4} \\ &\times \frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \Big) \\ &- \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5} a_{r_6} a_{r_5-r_6} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} \\ &\times \frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \Bigg) z^{r_1+5}. \end{aligned} \tag{52}$$

Consequently, the exact series of (1) is constructed as follows,

$$\begin{aligned} u(x,t) &= a_0 + a_1 (-x+t) + a_2 (-x+t)^2 + a_3 (-x+t)^3 + a_4 (-x+t)^4 \\ &+ \sum_{r_1=0}^{\infty} \Bigg(- \frac{(r_1+1)a_{r_1+1}}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\ &- \sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4} \Big(5a_{r_5} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} (r_2+2)(r_2+1) a_{r_2+2} \\ &\times \frac{(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)a_{r_1-2r_2+3}}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \\ &+ 5a_{r_5} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} a_{r_2+1} a_{r_1-2r_2+4} \\ &\times \frac{(r_1-2r_2+4)(r_1-2r_2+3)(r_1-2r_2+2)(r_1-2r_2+1)}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \Big) \\ &- \sum_{r_2=1}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_4}\sum_{r_6=0}^{r_5} a_{r_6} a_{r_5-r_6} a_{r_4-r_5} a_{r_3-r_4} a_{r_2-r_3} \\ &\times \frac{a_{r_1-r_2+5}(r_1-r_2+4)(r_1-r_2+3)(r_1-r_2+2)(r_1-r_2+1)}{a_0^5 (r_1+5)(r_1+4)(r_1+3)(r_1+2)(r_1+1)} \Bigg) (-x+t)^{r_1+5}, \end{aligned} \tag{53}$$

where $a_0 \neq 0$, a_1, a_2, a_3, a_4 are arbitrary constants. Eq. (44) may then be applied to obtain the rest of the constants. This systematic approach generates the solution (17).

Analogously, albeit tediously, it is easy to test for convergence of the power series solutions that correspond to the reduced equations derived above. Furthermore, these series solutions are also easily transformed into the original variables of the given equations. Due to the voluminous convergence procedure above, we omit the detailed testing for the other cases. In Figure 1 we display plots of selected solutions.

The reductions involving (17) and (40) include a travelling wave variable, based on the invariant obtained from the particular combination of symmetries X_1 and X_2, but these plots are of the truncated series solutions, and hence are, at best, approximate in nature.

We observe that Case A's truncated solutions grows faster and larger than Case B's truncated solution. It is difficult to make any further physical interpretations for the truncated solutions.

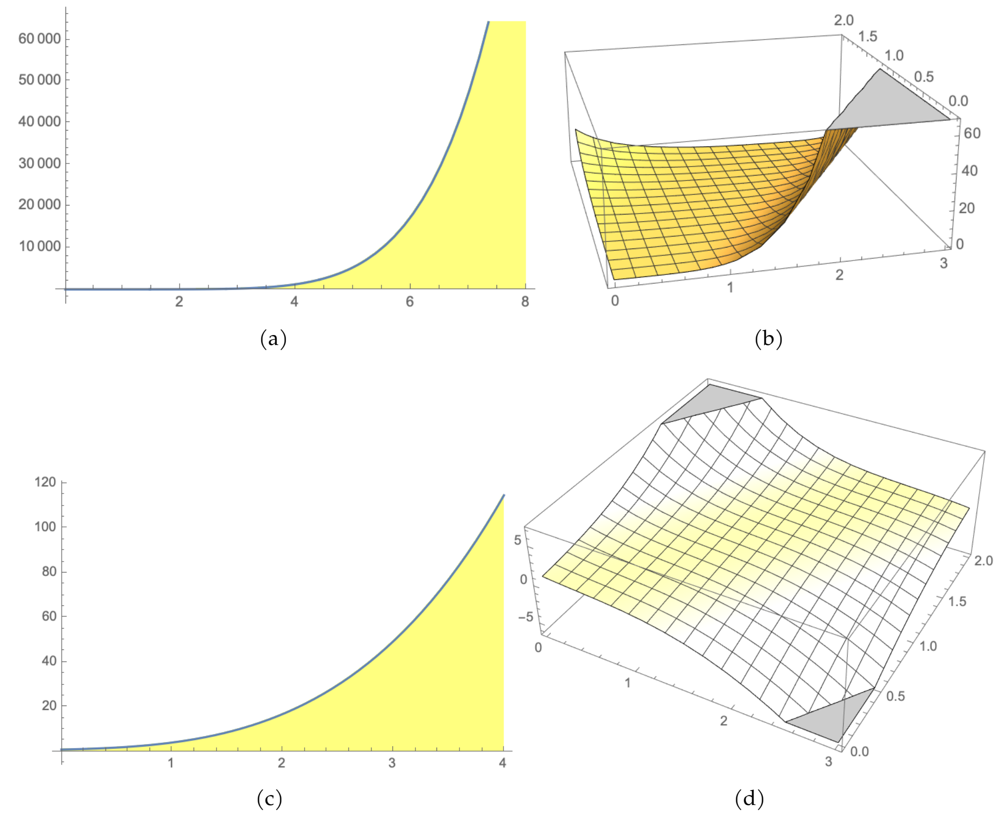

Figure 1: Graphical description of the analytical solutions with parameter values $\alpha = a_k = 1$: (a) Plot of Eq. (17); (b) 3D plot of Eq. (17); (c) Plot of Eq. (40); and (d) 3D plot of Eq. (40).

5 Conclusion

Nonlinear partial differential equations have remained a topic of great interest. In this regard, vast work has been done on the basis of various structures known as point symmetries. However, when the reduction process yields extremely nonlinear equations as well, the problem of finding solutions becomes more complicated and requires special tools. In such a situation, power series act as an active medium. Exploring solutions via power series is a simple technique but is hugely time consuming. Nonetheless, we have successfully demonstrated that symmetries combined with power series are a significant duo in unraveling solutions to difficult equations, and an added bonus is that there exists ways to address the associated convergence test of the resulting solutions. The obtained series solutions have been verified by Mathematica.

The novelty of this study is that this is the first time that a symmetry analysis of the fifth-order Fujimoto-Watanabe equation have been performed. Nonlinear equations are always a challenging topic of interest. Secondly, the power series approach has not been applied to these highly nonlinear models before, hence such solutions have been reported here only. There are many studies of the third-order models but not the fifth-order ones above. Therefore this study offers new knowledge of the invariant properties of the fifth-order models.

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Conflicts of Interest The authors declare no conflict of interest.

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